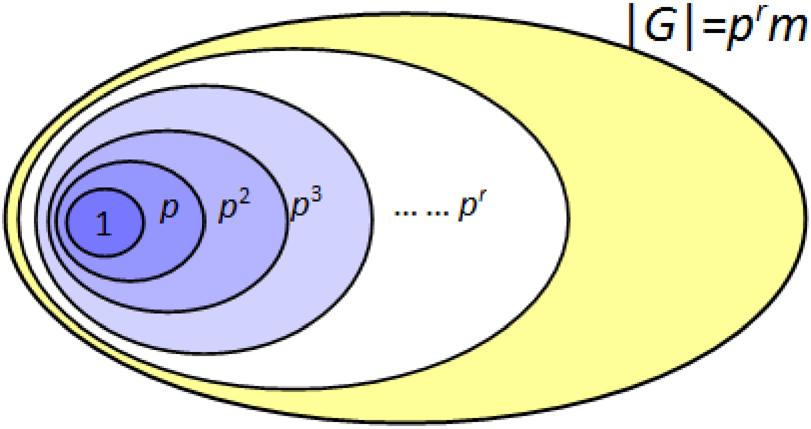
GROUPS:
THEIR PRESENTATIONS AND REPRESENTATIONS



VOLUME 1

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15th EDITION July 2022

These notes were prepared for students at Macquarie University in Australia but are freely available to anyone. However if you make use of them and are not a Macquarie University student it would be nice if you could email me at christopherdonaldcooper@gmail.com to let me know where you are from. And, if you are from outside of Australia perhaps you could send me a postcard of where you are from to pin up on my wall (Christopher Cooper, 31 Epping Avenue, EASTWOOD, NSW 2122, Australia).

I would like to acknowledge the assistance of Dr Ross Moore, Senior Lecturer at Macquarie University, for pointing out some errors and making some useful suggestions.

A second volume contains some more advanced topics, such as infinite abelian groups, soluble and nilpotent groups as well as some a few topics from my own research.

How does this differ from other introductions to group theory?

It is usual to introduce students to group theory via a course on groups, rings and fields rather than group theory in isolation. It is generally felt that students benefit from the parallel between the axiomatic development of groups and rings. Regrettably students don't see it that way and fail to be excited by a parallel between one unfamiliar theory and another.

These notes confine themselves to group theory alone with the aim of giving the student a wide experience there. By the time they learn about rings the parallel will be between the familiar and the unfamiliar.

But it might be objected that this will result in students learning more group theory than they need as an undergraduate. Certainly it would be a mistake to lead students into the technical depths of the subject, but there is a lot of room for breadth. What an undergraduate needs is a wide variety of mathematical experiences, and group theory can provide this.

The obvious experience is that of the development of an axiomatic system. Historically, group theory was the forerunner in this 'modern' approach to mathematics. And because there are only a few axioms, and a wide variety of different types of examples, students appreciate the experience of axiomatic abstraction within group theory much better than they do in many other areas, such as the theory of vector spaces.

Instead of merely illustrating the definitions and theorems, with a few well-chosen examples, we have attempted to inundate students with such a variety of examples that they come to value the theory as a way of organising this wide experience.

But mathematics is in the process of making another major change, and group theory is once again one of the leaders. Computation has moved from merely being a mathematical tool to being integrated into the theoretical development itself. Far from being a specialist area within group theory, computational group theory has become main-stream. And similar things are starting to happen in other branches of mathematics.

This has shifted the emphasis somewhat from the group axioms to the description of a group through generators and relations. Of course only finitely presented groups are accessible in this way and often groups arise independently of any particular presentation. But many important groups that do arise in applications do so as a presentation.

This justifies the inclusion of a chapter on the Todd-Coxeter algorithm. It is by no means a recent thing but it has become increasingly important lately. With the accompanying discussion of the unsolvability of the Word Problem (stated here but not proved) it also has the merit of introducing the student to the idea of the non-computable within a purely mathematical context. (Somehow to a mathematician the unsolvability of the Word Problem seems to hit home more than the unsolvability of the Halting Problem despite the two being equivalent.)

Another topic, which is normally considered too advanced for an introductory course on group theory, is representation theory. This is because a completely self-contained account really needs a fair knowledge of rings and modules. However the deeper aspects can be isolated into a single Fundamental Theorem of Characters ("the irreducible character form an orthonormal basis for the space of class functions") which can be taken on faith just as the Fundamental Theorem of Algebra is when complex numbers are first encountered.

From here everything can be built up with the help of a fair bit of standard linear algebra, thereby reinforcing the student's, often patchy, knowledge. The other benefit that has been found is that in constructing the character table for a group the student has to integrate a lot of elementary group theory. In particular the act of inducing characters from quotient groups in specific examples has been shown to cement the idea of a quotient group in a way that no amount of theory can.

These notes have been used quite effectively in conjunction with an individual project where each student receives the presentation of a specific non-abelian group. The task is first to use the Todd-Coxeter algorithm to produce a group table and then, largely by examining the quotient structure and using the technique of inducing characters from quotient groups, construct the character table for their group. It has been found that because each student is working with their own presentation, and because so much structure seems to miraculously grow from such a tiny ‘seed’ they are generally highly motivated and come away from the course with a much better grasp of the subject than with the traditional theorem-proof approach.

That is not to say that there are no proofs. Indeed nearly everything used here is proved It is just that the emphasis is a little more on *doing* than proving.

However the importance of proofs must not be underestimated. Students tend to hate proofs and often skip them. Indeed it is the fashion today for educators to take out many proofs, particularly from first year courses. To some extent this can be justified because in many areas of mathematics the proofs are not all that edifying.

With group theory however the proofs contain many important techniques and are an integral part of the experience. Remember that, although there are many applications of group theory, 99% of students will never use any group-theory facts in their employment. But the logic and problem-solving skills that they will develop in this course are very useful in many careers. So whether a student wants to become a program developer, an engineer, or even a barrister, they will have benefited from the training they will get here.

The theory that we will build up interlocks in a most remarkable way, with most proofs depending on previous theorems. One of the experiences, possibly only to be found in no other branch of mathematics, is where the reasoning is almost ‘forensic’. Little clues fit together to reach a conclusion, and the satisfaction gained isn’t so very different to the satisfaction a detective feels in solving a crime!

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